

QUANTUM MAPS AND AUTOMORPHISMS

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ABSTRACT. There are several inequivalent definitions of what it means to quantize a symplectic map on a symplectic manifold (M, ω) . One definition is that the quantization is an automorphism of a $*$ algebra associated to (M, ω) . Another is that it is unitary operator U_χ on a Hilbert space associated to (M, g) , such that $A \rightarrow U_\chi^* A U_\chi$ defines an automorphism of the algebra of observables. A yet stronger one, common in partial differential equations, is that U_χ should be a Fourier integral operator associated to the graph of χ . We compare the definitions in the case where (M, ω) is a compact Kähler manifold. The main result is a Toeplitz analogue of the Duistermaat-Singer theorem on automorphisms of pseudodifferential algebras, and an extension which does not assume $H^1(M, \mathbb{C}) = \{0\}$. We illustrate with examples from quantum maps.

1. INTRODUCTION

Much attention has been focussed recently on $*$ products on Poisson manifolds $(M, \{, \})$ (see, among others, [Kontsevich(1997), Cattaneo-Felder (2000), Karabegov-Schlichenmaier (2001), Tamarkin (1998), Reshetikhin-Takhtajan (1999), Etingof-Kazhdan (1996), Weinstein-Xu (1998)]). Such $*$ products are viewed as quantizing functions on M to an algebra of observables. This article is concerned with the related problem of quantizing symplectic maps χ on Kähler manifolds (M, ω) , a special case of the problem of quantizing Poisson maps. From the $*$ algebra viewpoint, it seems most natural to quantize such a symplectic map as an automorphism of a $*$ algebra associated to (M, ω) , specifically the (complete) symbol algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$ of Berezin-Toeplitz operators over (M, ω) . These symbol algebras are basic examples of abstract $*$ algebras arising in deformation quantization of Poisson manifolds (see [Boutet de Monvel 1(1999), Boutet de Monvel 2(1998), Boutet de Monvel 3 (1999), Charles (2003), Cattaneo-Felder (2000), Guillemin (1995), Schlichenmaier (1999), Schlichenmaier (1998)] for more on this aspect). But they carry more structure than bare $*$ algebras: the Toeplitz operator algebra \mathcal{T}^* of which $\mathcal{T}^*/\mathcal{T}^{-\infty}$ is the symbol algebra also comes with a representation as operators on a Hilbert space. In the Hilbert space setting, it is most natural to try to quantize a symplectic map χ as a unitary operator U_χ , and to induce the automorphism $U_\chi^* A U_\chi$ on \mathcal{T}^* . As will be explained below (see also [Zelditch (1997)]), it is not always possible to quantize a symplectic map this way. When possible, the quantization U_χ is an example of what is known as a quantum map in the literature of quantum chaos. Such quantum maps have also been the focus of much attention in recent years by a virtually disjoint group (see e.g. [de Bièvre -degli Esposti (1998), Keating (1991), degli Esposti-Graffi-Isola (1995), Hannay-Berry (1980), Marklof-Rudnick (2000), Zelditch (1997)]). The main purpose of this article is to contrast the different notions of quantizing symplectic maps, as they arise in Toeplitz $*$ algebras, partial differential equations and quantum chaos.

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Aside from its intrinsic interest, the relation between quantum maps and automorphisms of $*$ algebras has practical consequences in quantum chaos, i.e. in the relations between dynamical properties of χ and the eigenvalues/eigenfunctions of its quantization U_χ . In the physics literature of quantum chaos, quantum maps are studied through examples such as quantum kicked tops (on S^2), cat maps, rotors, baker's map and standard maps (on the 2-torus \mathbf{T}^2). Almost always, the quantizations are given as explicit unitary matrices U_N (depending on a Planck constant $1/N$) on special Hilbert spaces \mathcal{H}_N , often using some special representation theory, and no formal definition is given of the term 'quantum map'. The need for precise definitions is felt, however, as soon as one aims at quantizing maps which lie outside the range of standard examples. Even symplectic maps on surfaces of genus $g \geq 2$ count as non-standard, and only seem to have been quantized by the Toeplitz method discussed in this paper and in [Zelditch (1997)].

A further reason to study quantum maps versus automorphisms is to better understand obstructions to quantizations. It is often said that Kronecker translations

$$T_{\alpha,\beta}(x, \xi) = (x + \alpha, \xi + \beta), \quad (x, \xi) \in \mathbb{R}^{2n}/\mathbb{Z}^{2n},$$

and affine symplectic torus maps

$$f_\alpha(x, \xi) = (x + \xi, \xi + \alpha)$$

are not quantizable, for reasons explained in Proposition 2.2. Nevertheless, the paper [Marklof-Rudnick (2000)] proposes a quantization of such maps. Of course, the resolution of this paradox is that a weaker notion of quantization is assumed in [Marklof-Rudnick (2000)] than elsewhere, as will be explained below.

The implicit criterion (including that in [Marklof-Rudnick (2000)]) that U_N quantize a symplectic map χ is that the Egorov type formula

$$(1) \quad U_N^* Op_N(a) U_N \sim Op_N(a \circ \chi), \quad (N \rightarrow \infty)$$

hold for all elements $Op_N(a)$ of the algebra \mathcal{T}_N of observables, where χ is a symplectic map of (M, ω) . Postponing precise definitions, we see that the operative condition is that $U_N^* Op_N(a) U_N$ defines an automorphism of \mathcal{T}_N , at least to leading order. Here, our notation for observables and quantum maps are in terms of sequences as the inverse Planck constant N varies. We temporarily write \mathcal{T}^* for sequences $\{Op_N(a)\}$ of observables (with $\mathcal{T}^{-\infty}$ the sequences which are rapidly decaying in N), and $U \sim \{U_N\}$ for sequences of unitary quantum maps. We will soon give more precise definitions.

We now distinguish several notions of quantizing a symplectic map and make a number of assertions which will be justified in the remainder of the article.

- There is a geometric obstruction to quantizing a symplectic map χ as a Toeplitz quantum map $U_{\chi,N}$ on \mathcal{H}_N (see Definition 1.5 and Proposition 2.2). Kronecker translations and parabolic maps of the torus are examples of non-quantizable symplectic maps in the Toeplitz sense (see Propositions 5.1 and 5.3);
- There is no obstruction to quantizing a symplectic map as an automorphism of the Toeplitz symbol algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$ (see Theorem 1.6). For instance, Kronecker maps and cat maps are quantizable as automorphisms (see Propositions 5.2-5.4);
- Conversely, if $H^1(M, \mathbb{C}) = \{0\}$, then every order preserving automorphism of the symbol algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$ on M is induced by a symplectic map of (M, ω) (see Theorem 1.6 for this and for the case where $H^1(M, \mathbb{C}) \neq \{0\}$);

- There is an obstruction to ‘extending’ an automorphism α of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ as an automorphism of \mathcal{T}^* . In particular, there is an obstruction to inducing automorphisms α_N of the finite dimensional algebras of operators \mathcal{T}_N acting on \mathcal{H}_N (see Theorem 1.6). Again, Kronecker maps are examples (see §5).
- Any sequence U_N of unitaries on \mathcal{H}_N which defines an automorphism of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ must be a Toeplitz quantum map in sense of Definition 1.5 (i) (cf. [Boutet de Monvel 4 (1985), Zelditch (1997)]).
- Many of the key problems of quantum chaos, e.g. problems on eigenvalue level spacings or pair correlation, on ergodicity and mixing of eigenfunctions (etc.) concern only the spectral theory of the automorphism quantizing χ and not the unitary map per se (see §6).

1.1. The Toeplitz set-up. In order to state our results precisely, we need to specify the framework in which we are working. The framework of Toeplitz operators used in this paper is the same as in [Boutet de Monvel 1(1999), Boutet de Monvel 2(1998), Guillemin (1995), Bleher-Shiffman-Zelditch(2001), Shiffman-Zelditch, Zelditch (1997), Zelditch (1998)]. We briefly recall the notation and terminology.

Our setting consists of a Kähler manifold (M, ω) with $\frac{1}{2\pi}[\omega] \in H^1(M, \mathbb{Z})$. Under this integrality condition, there exists a positive hermitian holomorphic line bundle $(L, h) \rightarrow M$ over M with curvature form

$$c_1(h) = -\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log \|e_L\|_h = \omega,$$

where e_L is a nonvanishing local holomorphic section of L , and where $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h -norm of e_L . We give M the volume form $dV = \frac{1}{m!} \omega^m$.

The Hilbert spaces ‘quantizing’ (M, ω) are then defined to be the spaces $H^0(M, L^N)$ of holomorphic sections of $L^N = L \otimes \cdots \otimes L$. The metric h induces Hermitian metrics h_N on L^N given by $\|s^{\otimes N}\|_{h_N} = \|s\|_h^N$. We give $H^0(M, L^N)$ the inner product

$$(2) \quad \langle s_1, s_2 \rangle = \int_M h_N(s_1, s_2) dV \quad (s_1, s_2 \in H^0(M, L^N)),$$

and we write $|s| = \langle s, s \rangle^{1/2}$. We then define the Szegő kernels as the orthogonal projections $\Pi_N : \mathcal{L}^2(M, L^N) \rightarrow H^0(M, L^N)$, so that

$$(3) \quad (\Pi_N s)(w) = \int_M h_z^N(s(z), \Pi_N(z, w)) dV_M(z), \quad s \in \mathcal{L}^2(M, L^N).$$

Instead of dealing with sequences of Hilbert spaces, observables and unitary operators, it is convenient to lift them to the circle bundle $X = \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$, where L^* is the dual line bundle to L , and where h^* is the norm on L^* dual to h . Associated to X is the contact form $\alpha = -i\partial\bar{\partial}\rho|_X = i\bar{\partial}\rho|_X$ and the volume form

$$(4) \quad dV_X = \frac{1}{m!} \alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_M.$$

Holomorphic sections then lift to elements of the Hardy space $\mathcal{H}^2(X) \subset \mathcal{L}^2(X)$ of square-integrable CR functions on X , i.e., functions that are annihilated by the Cauchy-Riemann

operator $\bar{\partial}_b$ and are \mathcal{L}^2 with respect to the inner product

$$(5) \quad \langle F_1, F_2 \rangle = \frac{1}{2\pi} \int_X F_1 \overline{F_2} dV_X, \quad F_1, F_2 \in \mathcal{L}^2(X).$$

We let $r_\theta x = e^{i\theta} x$ ($x \in X$) denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial \theta}$. The S^1 action on X commutes with $\bar{\partial}_b$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^2(X)$ where $\mathcal{H}_N^2(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta} F(x)\}$. A section s_N of L^N determines an equivariant function \hat{s}_N on L^* by the rule

$$\hat{s}_N(\lambda) = (\lambda^{\otimes N}, s_N(z)) \quad , \quad \lambda \in L_z^*, \quad z \in M,$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. We henceforth restrict \hat{s} to X and then the equivariance property takes the form $\hat{s}_N(r_\theta x) = e^{iN\theta} \hat{s}_N(x)$. The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $\mathcal{H}_N^2(X)$. We refer to [Boutet de Monvel-Guillemin (1981), Boutet de Monvel-Sjöstrand (1976), Bleher-Shiffman-Zelditch(2001), Zelditch (1998)] for further background.

We now define the (lifted) Szegő kernel of degree N to be the orthogonal projection $\Pi_N : \mathcal{L}^2(X) \rightarrow \mathcal{H}_N^2(X)$. It is defined by

$$(6) \quad \Pi_N F(x) = \int_X \Pi_N(x, y) F(y) dV_X(y), \quad F \in \mathcal{L}^2(X).$$

The full Szegő kernel is the direct sum

$$(7) \quad \Pi = \bigoplus_{N=1}^{\infty} \Pi_N.$$

Following Boutet de Monvel-Guillemin [Boutet de Monvel-Guillemin (1981)], we then define:

Definition 1.1. *The $*$ algebra $\mathcal{T}^*(M)$ of Toeplitz operators of (M, ω) is the algebra of operators on $\mathcal{H}^2(X)$ of the form*

$$(8) \quad \Pi A \Pi = \bigoplus_{N=1}^{\infty} \Pi_N A_N \Pi_N, \quad A \in \Psi_{S^1}^*(X)$$

where $\Psi_{S^1}^*(X)$ is the algebra of pseudodifferential operators over X which commute with the S^1 action, and where

$$(9) \quad A_N = \int_{S^1} e^{iN\theta} A e^{-iN\theta} d\theta.$$

Here, $\mathcal{N} = \frac{1}{i} \frac{\partial}{\partial \theta}$ is the operator generating the S^1 action, whose eigenvalue in $\mathcal{H}_N^2(X)$ equals N .

Since the symbol of A is S^1 -invariant, Toeplitz operators of this kind possess an expansion

$$(10) \quad \Pi A \Pi \sim \mathcal{N}^s \sum_{j=0}^{\infty} \Pi a_j \Pi \mathcal{N}^{-j}$$

where $a_j \in C^\infty(M)$. We may also express it in the direct sum form

$$(11) \quad \Pi A \Pi = \sum_{N=1}^{\infty} \Pi_N a_N \Pi_N$$

where $a_N(z, \bar{z}) \in S_{scl}^s$ is a semiclassical symbol of some order s , i.e. admits an asymptotic expansion

$$(12) \quad a_N(z, \bar{z}) \sim N^s \sum_{j=0}^{\infty} N^{-j} a_j(z, \bar{z}), \quad a_j(z, \bar{z}) \in C^\infty(M)$$

in the sense of symbols. We define the *order* of a Toeplitz operator $\Pi A \Pi$ to be the order s of the symbol. The order defines a filtration of \mathcal{T}^* by spaces of operators \mathcal{T}^s of order $s \in \mathbb{R}$. See [Guillemin (1995)] for further background.

We also define ‘flat’ symbols $f(z, N) \in S_{scl}^{-\infty}$ is ~ 0 as functions satisfying $f = O(N^{-m})$ for all m . We then define $\mathcal{T}^{-\infty}$ to be the flat (or smoothing) Toeplitz operators (possessing a flat symbol). The following definition is important in distinguishing the automorphisms which concern us:

Definition 1.2. *The complete Toeplitz symbol algebra (or smooth Toeplitz algebra) is the quotient algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$.*

We often view $\bigoplus_{N=1}^{\infty} \Pi_N a_N \Pi_N$ as the sequence $\{\Pi_N a_N \Pi_N\}$ of operators on the sequence $\mathcal{H}_N \simeq H^0(M, L^N)$ of Hilbert spaces. The physicists’ notation for $\Pi_N a_N \Pi_N$ is $Op_N(a_N)$. Viewing symbols as sequences $\{a_N(z, \bar{z})\}$, we define the $*_N$ product by

$$(13) \quad \Pi_N a_N \Pi_N \circ \Pi_N b_N \Pi_N = \Pi_N a_N *_N b_N \Pi_N.$$

In the Appendix, we will describe the calculation of $a_N *_N b_N$ so that it will not seem abstract to the reader. We now introduce automorphisms:

Definition 1.3. *An order preserving automorphism α of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ is an automorphism which preserves the filtration $\mathcal{T}^s/\mathcal{T}^{-\infty}$. We denote the algebra of such automorphisms by $Aut_o(\mathcal{T}^*/\mathcal{T}^{-\infty})$.*

It is important to distinguish:

- Order preserving automorphisms of \mathcal{T}^* which preserve $\mathcal{T}^{-\infty}$;
- Order preserving automorphisms of the symbol algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$.

Since elements of \mathcal{T}^* and of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ commute with the S^1 action, either kind of automorphism satisfies:

$$(14) \quad \alpha\left(\bigoplus_{N=1}^{\infty} \Pi_N a_N \Pi_N\right) \sim \bigoplus_{N=1}^{\infty} \Pi_N b_N \Pi_N,$$

where b_N is a semiclassical symbol of the same order as a_N . In the case of automorphisms of \mathcal{T}^* , we can conclude that $\alpha(\Pi_N a_N \Pi_N) = \Pi_N b_N \Pi_N$ and that α induces automorphisms α_N of the finite dimensional algebras \mathcal{T}_N for fixed N . However, for automorphisms of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ in general, $\alpha(\Pi_N a_N \Pi_N)$ is not even defined since $\Pi_N a_N \Pi_N \in \mathcal{T}^{-\infty}$. To put it another way, we cannot uniquely represent an element of the finite dimensional algebra as $\Pi_N a_N \Pi_N$ although we can uniquely represent elements of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ this way.

1.1.1. Covariant and Contravariant symbols. Let $\Pi_N a \Pi_N$ be a Toeplitz operator. By the *contravariant symbol* of $\Pi_N a \Pi_N$ is meant the multiplier a_N . By the *covariant symbol* of an

operator F is meant the function

$$(15) \quad \hat{f}(z, \bar{z}) = \frac{\langle F\Phi_N^z, \Phi_N^w \rangle}{\langle \Phi_N^z, \Phi_N^w \rangle} \Big|_{z=w} = \frac{\Pi_N F \Pi_N(z, z)}{\Pi_N(z, z)}.$$

where

$$(16) \quad \Phi_N^w(z) = \frac{\Pi_N(z, w)}{\|\Pi_N(\cdot, z)\|}$$

is the L^2 -normalized ‘coherent state’ centered at w . When $F = \Pi_N a \Pi_N$ we get

$$(17) \quad \hat{a}(z, \bar{z}) = \frac{\Pi_N a \Pi_N(z, z)}{\Pi_N(z, z)}.$$

We use the notation $I_N(a) = \hat{a}$ for the linear operator (the Berezin transform) which takes the contravariant symbol to the covariant symbol (see [Reshetikhin-Takhtajan (1999)] for background).

1.2. Statement of results. Let us now consider the senses in which we can quantize symplectic maps in our setting. The first sense is that of quantizations of symplectic maps as Toeplitz Fourier integral operators. The definition is as follows.

Definition 1.4. *Suppose that the symplectic map χ of (M, ω) lifts to (X, α) as a contact transformation $\tilde{\chi}$. By the Toeplitz Fourier integral operator (or quantum map) defined by χ we mean the operator,*

$$U = \bigoplus_N U_{\chi, N}, \quad U_{\chi, N} = \Pi_N T_\chi \sigma_N \Pi_N$$

where $T_\chi : \mathcal{L}^2(X) \rightarrow \mathcal{L}^2(X)$ is the translation $T_\chi(f) = f \circ \tilde{\chi}^{-1}$ and where σ_N is a symbol designed to make $U_{\chi, N}$ unitary. (Such a symbol always exists [Zelditch (1997)]).

We now distinguish several notions of quantizing a symplectic map.

Definition 1.5. *Let χ be a symplectic map of (M, ω) . In descending strength, we say that:*

- (a) χ is quantizable as a Toeplitz quantum map (or Toeplitz Fourier integral operator) if it lifts to a contact transformation $\tilde{\chi}$ of (X, α) . The quantization is then that of Definition 1.4;
- (b) χ is quantizable as an automorphism of the full observable algebra if there exists an automorphism α of \mathcal{T}^* satisfying (1);
- (c) χ is quantizable as an automorphism of the symbol algebra if there exists an automorphism α of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ satisfying (1);

By descending strength, we mean that quantization in a sense above implies quantization in all of the following senses. The automorphisms above are *order-preserving* in the sense that the order of $\alpha(\Pi A \Pi)$ is the same as the order of $\Pi A \Pi$. Henceforth, all automorphisms will be assumed to be order-preserving.

We now explain the relations between these notions of quantization. We are guided in part by the analogous relations between quantizations of symplectic maps (of cotangent bundles) and automorphisms of the symbol algebra $\Psi^*/\Psi^{-\infty}$ of the algebra of pseudodifferential operators, as determined by Duistermaat-Singer in [Duistermaat-Singer (1976), Duistermaat-Singer (1975)]. Their main result was that, if $H^1(S^*M, \mathbb{C}) = \{0\}$, then every order preserving automorphism of $\Psi^*/\Psi^{-\infty}$ is either conjugation by an elliptic Fourier

integral operator associated to the symplectic map or a transmission. We prove an analogous theorem for Toeplitz operators and also extend it to the case where the phase space is not simply connected.

To state the results, we need some notation. We denote the universal cover of (M, ω) by \tilde{M} and denote the group of deck transformations of the natural cover $p : \tilde{M} \rightarrow M$ by Γ . We lift all objects on M to \tilde{M} under p . We denote by T_γ the unitary operator of translation by γ on $\mathcal{L}^2(\tilde{M})$. We also denote by \mathcal{T}_Γ^* the algebra of Γ -invariant Toeplitz operators on \tilde{M} . It is important to understand that \mathcal{T}_Γ^* is not isomorphic to the algebra of Toeplitz operators on M since there are non-trivial (smoothing) operators which act trivially on automorphic (periodic) functions. In other words, the representation of \mathcal{T}_Γ^* on automorphic sections has a kernel, which we denote by \mathcal{K}_Γ , and $\mathcal{T}^*(X) \simeq \mathcal{T}_\Gamma^*(\tilde{X})/\mathcal{K}_\Gamma$. Automorphisms which descend to the finite Toeplitz algebras are precisely those which preserve the subalgebra \mathcal{K}_Γ . For further discussion, we refer to §4.

Theorem 1.6. *With the above notation, we have:*

- (0) (Essentially known) A symplectic map of (M, ω) lifts to a contact transformation of (X, α) and hence defines a Toeplitz quantum map if and only if it preserves holonomies of all closed curves of M . (See Proposition 2.2 of §2).
- (i) Any symplectic map of any compact Kähler manifold (M, ω) , is quantizable as an automorphism of the algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$ of smooth Toeplitz operators over M ;
- (ii) Suppose that $H^1(M, \mathbb{C}) = \emptyset$. Then any order-preserving automorphism of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ is given by conjugation with a Toeplitz Fourier integral operator on M associated to a symplectic map χ of (M, α) . (The map lifts to a contact transformation of (X, α) by (0)).
- (iii) Suppose $H^1(M, \mathbb{C}) \neq \emptyset$. Then to each automorphism of $\mathcal{T}^*/\mathcal{T}^{-\infty}$ there corresponds a symplectic map χ of (M, ω) and a Toeplitz Fourier integral operator (Definition 1.5) U_χ on the universal cover \tilde{M} which satisfy $T_\gamma^* U_\chi T_\gamma = M_\gamma U_\chi$, where M_γ is a central operator. The automorphism $A \rightarrow U_\chi^* A U_\chi$ is Γ -invariant, and defines an order-preserving automorphism of the algebra \mathcal{T}_Γ^* which induces α on the Γ -invariant symbol algebra $\mathcal{T}^*/\mathcal{T}^{-\infty}$.
- (iv) Let $\mathcal{K}_\Gamma = \ker \rho_\Gamma$, where ρ_Γ is the representation of \mathcal{T}_Γ^* on Γ -automorphic functions on \tilde{M} . If α preserves \mathcal{K}_Γ , then it induces an order-preserving automorphism on $\mathcal{T}^*(M)$ and hence on the finite rank observables $Op_N(a)$ on \mathcal{H}_N .

We separate the proof into the cases $H^1(M, \mathbb{C}) = \{0\}$ in §3 and $H^1(M, \mathbb{C}) \neq \{0\}$ in §4. The latter case is very common in the physics literature on quantum maps. The difference between order preserving automorphisms of \mathcal{T}^* and $\mathcal{T}^*/\mathcal{T}^{-\infty}$ is very significant, and only the former automorphisms are quantum maps in the physics sense. For instance, as will be seen in §2, Kronecker maps and affine symplectic maps are quantizable as automorphisms of the symbol algebra, but do not lift to contact transformations of X , do not preserve the kernel \mathcal{K}_Γ and are therefore not automorphisms of \mathcal{T}^* . Regarding (iv), it is not clear to us whether this operator condition is equivalent to the holonomy-preservation condition in (0).

As a corollary, we prove a result which indicates that the physicists' quantum maps are necessarily Toeplitz quantum maps once they are conjugated to the complex (Bargmann) picture.

Corollary 1.7. *If $\{U_N\}$ is a sequence of unitary operators on \mathcal{H}_N and if $\bigoplus U_N$ defines an order preserving automorphism α of $\mathcal{T}^*/\mathcal{T}^{-\infty}$, then $\{U_N\}$ must be a Toeplitz Fourier integral operator associated to a quantizable symplectic map in the sense of Definition 1.4 .*

As a gauge of our definitions, let us reconsider the Marklof-Rudnick quantizations mentioned above of Kronecker maps, parabolic maps and other ‘non-quantizable’ maps [Marklof-Rudnick (2000)]. They define a sequence $\{U_N\}$ of unitary operators on \mathcal{H}_N satisfying the leading order condition (1), but not to any lower order. Hence, conjugation $U_\chi Op_N(a) U_\chi^*$ of an observable in \mathcal{T}^* by their quantum map is no longer an observable, i.e. it is not an element of \mathcal{T}^* . Rather, its Toeplitz symbol only possesses a one term asymptotic expansion and is not a classical symbol. Hence it need not correspond to a quantizable symplectic map.

In addition to Theorem 1.6, we discuss a related issue revolving around the quantum maps versus automorphisms distinction: From the viewpoint of quantum chaos, the main interest in the quantum maps $U_{\chi,N}$ lies in their spectral theory and its relation to the dynamics of χ . This is only well-defined when the associated symplectic map is quantizable in the strong sense as a sequence of unitary operators on \mathcal{H}_N . As stated in the corollary, the symplectic map must then lift to a contact transformation. In the last section §6, we point out that even when the symplectic map is quantizable as a unitary operator, it is often the automorphism it induces which is most significant in quantum chaos. That is, much of the spectral theory in quantum chaos concerns the spectrum of the automorphism induced by $U_{\chi,N}$ rather than the spectrum of $U_{\chi,N}$ itself.

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2. TOEPLITZ QUANTIZATION OF SYMPLECTIC MAPS

In this section, we consider the quantization of symplectic maps as Toeplitz Fourier integral operators. In some sense, the material in this section is known, but it seems worthwhile to recall the material and to complete some of the arguments.

Suppose that $\chi : (M, \omega) \rightarrow (M, \omega)$ is a symplectic diffeomorphism. There are several equivalent ways to state the condition that χ is quantizable. The most ‘geometric’ one is the following:

Definition 2.1. *χ is quantizable if χ lifts to a contact transformation $\tilde{\chi}$ of (X, φ) , i.e. a diffeomorphism of X such that $\tilde{\chi}^*\varphi = \varphi$.*

Equivalently, χ lifts to an automorphism of each power L^N of the prequantum complex line bundle. It is said to be linearizable in algebraic geometry.

Let us consider the obstruction to lifting a symplectic map. We follow in part the discussion in [Guillemin-Sternberg (1977)], p. 220. The key notion is that χ preserve the holonomy map of the connection 1-form α . Recall that the horizontal sub-bundle $H \subset TX$ of the connection is defined by $H_x = \ker \alpha_x = \{v \in T_x X : \alpha_x(v) = 0\}$. The holonomy map

$$H : \Lambda \rightarrow U(1), \quad H(\gamma) = e^{i\theta_\gamma}$$

from the free loop space defined by horizontally lifting a loop $\gamma : [0, 1] \rightarrow M$ to $\tilde{\gamma} : [0, 1] \rightarrow X$ and expressing $\tilde{\gamma}(1) = e^{i\theta_\gamma} \tilde{\gamma}(0)$. We say that χ is holonomy-preserving if

$$(18) \quad H(\chi(\gamma)) = H(\gamma), \quad \forall \gamma \in \Lambda.$$

If the loop is contained in the domain of a local frame $s : U \rightarrow X$, then

$$(19) \quad H(\gamma) = \exp(2\pi i \int_{\gamma} s^* \alpha).$$

If $\gamma = \partial\sigma$, then $\int_{\gamma} s^* \alpha = \int_{\sigma} \omega$. It follows (see [Guillemin-Sternberg (1977)]) that symplectic map preserves the holonomy around such homologically trivial loops. Hence it is sufficient to consider the map

$$(20) \quad H_{\chi} : H^1(M, \mathbb{Z}) \rightarrow U(1), \quad H_{\chi}(\gamma) = H(\gamma)^{-1} H(\chi(\gamma)) = e^{i(\theta_{\gamma} - \theta_{\chi(\gamma)})}$$

Proposition 2.2. *A symplectic map χ of a symplectic manifold (M, g) lifts to a contact transformation of the associated prequantum S^1 bundle (X, α) if and only if $H_{\chi} \equiv 1$, the trivial representation.*

Proof. Suppose that χ lifts to $\tilde{\chi} : X \rightarrow X$ as a contact transformation. Let $\gamma \in \Lambda$ and let $\tilde{\gamma}$ be a horizontal lift of γ . Then $\tilde{\chi}(\tilde{\gamma})$ is a horizontal lift of $\chi(\gamma)$. Obviously, $\tilde{\gamma}(1) = e^{i\theta_{\gamma}} \tilde{\gamma}(0)$ implies $\tilde{\chi} \circ \tilde{\gamma}(1) = e^{i\theta_{\gamma}} \tilde{\chi} \circ \tilde{\gamma}(0)$, so $H = H \circ \chi$.

Conversely, suppose that $H_{\chi} = 1$. We then define $\tilde{\chi}$ by lifting χ along paths. We fix a basepoint $x_0 \in MX$ and define $\tilde{\chi}$ on the orbit $S^1 \cdot x_0$ by fixing $\tilde{\chi}(x_0)$ to be a chosen basepoint on $\pi^{-1}(\chi(\pi(x_0)))$ and then extending by S^1 invariance. We now consider horizontal paths $x(t) : [0, 1] \rightarrow X$ from x_0 . At least one horizontal path exists from x_0 to any given point since the curvature is positive (Chow's theorem). We define $\tilde{\chi}(x(t))$ to be the horizontal lift of $\chi(\pi x(t))$ to $\tilde{\chi}(x_0)$. To see that this is well-defined, we must prove independence of the path. So let $x_1(t), x_2(t)$ be two horizontal paths from x_0 to $x_1(1)$. Thus, there is trivial holonomy of the loop defined by x_1 followed by x_2^{-1} (i.e. the backwards path to x_2). Now project each path, apply χ , and horizontally lift. This defines a horizontal lift of the loop formed by the projected curves $\chi \circ \pi \circ x_j(t)$ ($j = 1, 2$). It has trivial holonomy if χ is holonomy preserving. It follows that the horizontal lifts must agree at $t = 1$. □

It follows that χ always lifts to a contact transformation if M is simply connected. Hence, if we lift χ first to the universal cover \tilde{M} of M , then this further lifts to \tilde{X} as a contact transformation. We verify this in another way, since we will use it in §4:

Proposition 2.3. *Let χ be a symplectic map of \tilde{M} . Then there exists a unique (up to one scalar) lift $\tilde{\chi}$ of χ to \tilde{X} such that*

$$\pi \tilde{\chi} = \chi \pi$$

where $\pi : \tilde{X} \rightarrow \tilde{M}$ is the S^1 -fibration.

Proof. The fact that χ can be lifted to \tilde{X} is obvious since $\tilde{X} \simeq \tilde{M} \times S^1$. The key point is that the map can be lifted as contact transformations. Any lift which commutes with the S^1 action has the form

$$(21) \quad \tilde{\chi} \cdot (z, e^{i\theta}) = (\chi(z), e^{i\theta + \varphi_{\chi}(\theta, z)}).$$

The contact form on \tilde{X} is the connection 1-form $\tilde{\alpha}$ of the hermitian line bundle over \tilde{X} . In local symplectic coordinates (x, ξ) on \tilde{M} it has the form

$$\tilde{\alpha} = \frac{1}{2}(\xi dx - x d\xi) - d\theta.$$

Since $\chi^*(xd\xi - \xi dx) - (xd\xi - \xi dx)$ is closed on \tilde{M} , and since \tilde{M} is simply connected, there exists a function $f_\chi \in C^\infty(\tilde{M})$ such that

$$\tilde{\chi}^*(xd\xi - \xi dx) - (xd\xi - \xi dx) = df_\chi(x, \xi).$$

Using the product structure, we have that

$$\varphi_\chi(x, \xi, \theta) = f_\chi(x, \xi)$$

defines a lift satisfying $\tilde{\chi}^*(\tilde{\alpha}) = \tilde{\alpha}$, as desired.

Regarding uniqueness: the only flexibility in the lift is in the choice of f_χ , which is defined up to a constant. The constant can be fixed by requiring that $f_\chi(0, 0) = 0$. □

There is a weaker condition which has come up in some recent work (cf. [Marklof-Rudnick (2000)]): let us say that χ is quantizable at level N if χ lifts to an automorphism of the bundle L^N . Often a map is quantizable of level N along an arithmetic progression $N = kN_0, k = 1, 2, 3, \dots$ of powers although it is not quantizable for all N . In geometric terms, this simply means that χ fails to lift as a contact transformation of X but does lift as a contact transformation of X/\mathbb{Z}_N where $\mathbb{Z}_N \subset S^1$ is the group of N th roots of unity. In everything that follows, the stated results have analogous for this modified version of quantization.

3. PROOF OF THEOREM 1.6 IN THE CASE $H^1(M, \mathbb{C}) = \emptyset$

We first prove that if $H^1(M, \mathbb{C}) = \emptyset$, then every automorphism is given by conjugation with a Toeplitz Fourier integral operator. In this case, we may identify maps on M which S^1 invariant maps on X . We emphasize that we are not considering the most general Toeplitz operators $\Pi A \Pi$ with $A \in \Psi^*(X)$ but only the S^1 -invariant operators whose symbols lie in $C^\infty(M)$. The proof is modelled on that of Duistermaat-Singer [Duistermaat-Singer (1976)], but has several new features due to the holomorphic setting. In some respects the proof is simpler, since there are no transmission automorphisms, and there are natural identifications between symbols of different orders. However in some respects it is more complicated, and also we must be careful about using contravariant versus covariant symbols.

We begin with:

Lemma 3.1. *Suppose that $H^1(M, \mathbb{C}) = \emptyset$ and that ι is an order-preserving automorphism of $\mathcal{T}^\infty/\mathcal{T}^{-\infty}$. Then ι is equal to conjugation by a Toeplitz Fourier integral operator in the sense of Definition 1.4.*

Proof. Since ι is order preserving it induces automorphisms on the quotients of the filtered algebra \mathcal{T}^* .

We first consider $\mathcal{T}^0/\mathcal{T}^{-1}$. The map to contravariant symbols defines an identification with $C^\infty(M)$. Thus ι induces an automorphism of $C^\infty(M)$, viewed as an algebra of contravariant symbols under multiplication. The maximal ideal space of $C(M)$ equals M , hence ι induces a map χ on M such that $\iota(p) = p \circ \chi$. Precisely as in [Duistermaat-Singer (1976)] one verifies that χ is a smooth diffeomorphism of M .

Now consider the quotients $\mathcal{T}^m/\mathcal{T}^{m-1}$. They are simply N^m times $\mathcal{T}^0/\mathcal{T}^{-1}$, so for any m $\iota(p) = p \circ \chi$ for $p \in \mathcal{T}^m/\mathcal{T}^{m-1}$. This step is simpler than in the pseudodifferential case, and as a result certain steps carried out in [Duistermaat-Singer (1976)] are unnecessary here.

Now let $n = 1$, so that $\mathcal{T}^1/\mathcal{T}^0$ is a Lie algebra under commutator bracket. The principal symbol is an isomorphism of the quotient algebra to the Poisson algebra $(M, \{, \})$ defined by the symplectic form ω . Since ι is an automorphism of the quotient algebra, we have

$$\{a \circ \chi, b \circ \chi\} = \{a, b\} \circ \chi,$$

hence χ is a symplectic map of (M, ω) . This step is also simpler than in [Duistermaat-Singer (1976)], and we see that no transmissions arise as possible automorphisms.

By Proposition 2.3, χ lifts to a contact transformation $\tilde{\chi}$ of X .

3.0.1. *Symbol preserving automorphisms.* Now let $A_N^{-1} = \Pi_N a T_\chi^{-1} \Pi_N$ denote any Toeplitz quantization of $\tilde{\chi}^{-1}$. It follows that $\alpha(P) = A_N \iota(P) A_N^{-1}$ is an automorphism of $\mathcal{T}^\infty/\mathcal{T}^{-\infty}$ which preserves principal contravariant symbols in the sense of §1.1.1. We now prove that any such automorphism is given by conjugation with a Toeplitz multiplier of some order s .

Thus, let j be a principal contravariant symbol preserving automorphism. Let $P \in \mathcal{T}^m$. Since $j(P) - P \in \mathcal{T}^{m-1}$, we get an induced map

$$(22) \quad \beta_m : C^\infty(M) \rightarrow C^\infty(M), \quad \beta_m(a) = j(\Pi_N a \Pi_N) - \Pi_N a \Pi_N.$$

Then $\beta = \beta_m$ is a derivation in two ways:

$$(23) \quad \begin{aligned} (i) \quad & \beta(p \cdot q) = \beta(p) \cdot q + p \cdot \beta(q) \\ (ii) \quad & \beta(\{p, q\}) = \{\beta(p), q\} + \{p, \beta(q)\}. \end{aligned}$$

Now any derivation in the sense of (i) is given by differentiation along a vector field V . Since V commutes with Poisson bracket, it must be a symplectic vector field. Since $\omega(V, \cdot) = \beta$ is a closed 1-form, there exists a local Hamiltonian H for V . Under our assumption that $H^1(M) = \{0\}$, the Hamiltonian is global, so $V = \Xi_H$ for some global H . Thus we have:

$$(24) \quad \beta_m(a) = i\{a, H_{\log b}\}$$

for some $b_m \in C^\infty(M)$. Thus, j may be represented by $j(P) = B^{-1}PB$ for $P \in \mathcal{T}^m$, with $B = \Pi_N e^{ib} \Pi_N$ ($b = b_m$). Because of the natural identification of $S_{scl}^m \equiv N^m S_{scl}^0$, we find that $b_m = b$ is the same for all m .

By composing automorphisms, we now have an automorphism j_2 such that

$$j_2(\Pi_N a_N \Pi_N) - \Pi_N a_N \Pi_N \in \mathcal{T}^{m-2}, \quad a_N \in S_{scl}^m.$$

We find as above that $j_2(\Pi_N a_N \Pi_N) - \Pi_N a_N \Pi_N = \beta_2(a_N)$ where β_2 is a derivation, hence $\beta_2 = N^{-1}\{\log b_{-1}, \cdot\}$ for some b_{-1} . Therefore,

$$(25) \quad \Pi_N e^{-iN^{-1}b_{-1}} \Pi_N j_2(P) \Pi_N e^{iN^{-1}b_{-1}} \Pi_N - P \in \mathcal{T}^{m-2}, \quad \forall P \in \mathcal{T}^m.$$

Proceeding in this way, we get an element $b_N \in S_{scl}^0$ such that

$$(26) \quad \Pi_N e^{-ib_N} \Pi_N j(P) \Pi_N e^{ib_N} \Pi_N - P \in \mathcal{T}^{-\infty}, \quad \forall P \in \mathcal{T}^m.$$

This completes the proof of the Lemma. □

3.0.2. *Conclusion of proof when $H^1(M, \mathbb{C}) = \emptyset$.* Lemma 3 of [Duistermaat-Singer (1976)] is an abstract result which says that automorphisms of Frechet spaces satisfying a certain density condition are always given by conjugation. The density condition is easy to prove, so we omit the proof. The result is:

Let ι denote an automorphism of \mathcal{T}^ acting on $H^\infty(M)$. Then $\iota(P) = A^{-1}PA$, where $A : H^\infty(M) \rightarrow H^\infty(M)$ is an invertible, continuous linear map, determined uniquely up to multiplicative constant.*

Thus, $i(P) = A^{-1}PA$ for all $P \in \mathcal{T}^*$ and also, by Lemma 1, there exists an elliptic Toeplitz Fourier integral operator B such that $i(P) \equiv B^{-1} \circ P \circ B$ for all $P \in \mathcal{T}^\infty/\mathcal{T}^{-\infty}$. Let $E = A \circ B^{-1}$. Then $[E, P] \in \mathcal{T}^{-\infty}$ for all $P \in \mathcal{T}^\infty/\mathcal{T}^{-\infty}$. In particular, $[E, P] \in \mathcal{T}^{-\infty}$ for all $P = \{\Pi_N a \Pi_N\}$, $a \in C^\infty(M)$.

In place of [Duistermaat-Singer (1976)], Lemma 4, we use

Lemma 3.2. *Let E be an operator on \mathcal{H}^2 such that $[E, \Pi a \Pi] \in \mathcal{T}^{-\infty}$ for all $a \in C^\infty(M)$. Then there exists a constant c such that $E = c\Pi + R$, where R is a smoothing operator.*

Proof. It is sufficient to prove the statement for all a supported in a given S^1 -invariant open set $U \subset X$. We can then use a partition of unity to prove the result for all a . We use the notation $A \sim_U B$ to mean that A, B are defined on U and their difference is a smoothing operator on U .

In a sufficiently small open set $U \subset X$, there exists a Fourier integral operator $F : L^2(X) \rightarrow L^2(\mathbb{R}^n)$ associated to a contact transformation φ such that $\Pi \sim_U F \Pi_0 F^*$ modulo smoothing operators, where Π_0 is the model Szegő kernel discussed in [Boutet de Monvel-Sjöstrand (1976), Boutet de Monvel-Guillemin (1981)], namely the orthogonal projection onto the kernel of the annihilation operators $D_j = \frac{1}{i}(\partial/\partial y_j + y_j |D_t|)$ on $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$. Furthermore, it is proved in [Boutet de Monvel-Sjöstrand (1976), Boutet de Monvel-Guillemin (1981)] that there exists a complex Fourier integral operator $R_0 : L^2(\mathbb{R}^q) \rightarrow L^2(\mathbb{R}^n)$ such that $R_0^* R_0 = I$, $R_0 R_0^* = \Pi_0$. Moreover for any pseudodifferential operator A on X , there exists a pseudodifferential operator Q on \mathbb{R}^q so that $\Pi_0 A \Pi_0 \sim R_0 Q R_0^*$. Transporting R_0 to X by F , we obtain a complex Fourier integral operator $R : L^2(\mathbb{R}^q) \rightarrow L^2(X)$ so that $RR^* \sim_U \Pi$, $R^* R \sim_{\varphi(U)} I$ and so that $\Pi a \Pi \sim_U R Q R^*$. Then $[E, \Pi a \Pi] \sim_U [E, R^* Q R]$ and we may rewrite the condition on E as:

$$(27) \quad [E, R^* Q R] \in \mathcal{T}^{-\infty}(X), \quad \forall Q \in \Psi^0(\mathbb{R}^q).$$

This is equivalent to

$$(28) \quad [R^* E R, Q] \in \Psi^{-\infty}(X), \quad \forall Q \in \Psi^0(\mathbb{R}^q).$$

We then apply Beals' characterization of pseudodifferential operators: $P \in \Psi^k(\mathbb{R}^m)$ if and only if for all $\{j_i, k_\ell\}$,

$$ad(x_{j_1}) \cdots ad(x_{j_r}) ad(D_{x_{k_1}}) \cdots ad(D_{x_{k_s}}) P \in H^{s+r}(\mathbb{R}^m) \rightarrow H^s(\mathbb{R}^m)$$

is bounded. Here, $ad(L)P$ denotes $[L, P]$. It follows first that $R^* E R \in \Psi^0$, and easy symbol calculus shows that the complete symbol of $R^* E R$ is constant. Hence, $R^* E R = I + S$ where S is a smoothing operator. Applying R on the left and R^* on the right concludes the proof. \square

Remark: It is in the step in §3.0.1 that the distinction between symplectic maps of M and contact transformations of X enters. Ultimately it is this step which leads to Corollary 1.7. We also note that the proof above is rather different from that in [Duistermaat-Singer (1976)].

4. $H^1(M, \mathbb{C}) \neq \emptyset$

The problems with quantizing symplectic maps on M are all due to the fundamental group $\pi_1(M)$ or more precisely $H_1(M, \mathbb{C})$. We solve them by passing to the universal cover \tilde{M} . In this section, we relate Toeplitz operators on M and \tilde{M} .

4.1. Toeplitz operators on the universal cover. Since we are comparing algebras and automorphisms on covers to those on a quotient, we begin with the abstract picture as discussed in [Gromov-Henkin-Shubin (1998)]. We then specialize it to algebras of Toeplitz operators.

4.1.1. Abstract theory. Suppose that $p : \tilde{X} \rightarrow X$ is a covering map of a compact manifold X , and denote its deck transformation group by Γ . We regard Γ as acting on the left of \tilde{X} . We would like to compare operators on \tilde{X} and operators on X . To gain perspective, we start with the large von Neumann algebra \mathcal{B}_Γ of all bounded operators on $\mathcal{L}^2(\tilde{X})$ which commute with Γ . Later we specialize to the Toeplitz algebra which is our algebra of observables.

The Schwartz kernel of such an operator satisfies $B(\gamma x, \gamma y) = B(x, y)$. If we denote by \mathcal{D} a fundamental domain for Γ , then there exists an identification

$$(29) \quad \mathcal{L}^2(\tilde{X}) \simeq \mathcal{L}^2(\Gamma) \otimes \mathcal{L}^2(\mathcal{D}) = \mathcal{L}^2(\Gamma) \otimes \mathcal{L}^2(X).$$

Elements of $\mathcal{L}^2(\Gamma) \otimes \mathcal{L}^2(X)$ can be viewed as functions $f(\gamma, x)$ on $\Gamma \times \mathcal{D}$. The unitary isomorphism is defined by

$$\varphi \in \mathcal{L}^2(\tilde{X}) \rightarrow f_\varphi(\gamma, x) = \varphi(\gamma \cdot x).$$

Note that both left translation L_γ and right translation R_γ by γ act on this space, namely

$$L_\gamma f(\alpha, x) = f(\gamma\alpha, x), \quad R_\gamma f(\alpha, x) = f(\alpha\gamma, x).$$

We may regard \mathcal{B}_Γ as bounded operators commuting with all L_γ . The isomorphism (29) induces an algebra isomorphism

$$(30) \quad \mathcal{B}_\Gamma \simeq \mathcal{R} \otimes \mathcal{B}(X),$$

where $\mathcal{B}(X)$ is the algebra of bounded operators on X and where \mathcal{R} is the algebra generated by right translations R_γ on $L^2(\Gamma)$.

So far we have been considering operators on $\mathcal{L}^2(\tilde{X})$. The corresponding algebra \mathcal{B}_Γ is much larger than $\mathcal{B}(X)$. To make the connection to $\mathcal{L}^2(X)$ tighter, we need to consider the space $\mathcal{L}_\Gamma^2(\tilde{X})$ of Γ -periodic functions on \tilde{X} . The natural Hilbert space structure is to define $\|f\|_\Gamma^2 = \int_{\mathcal{D}} |f(x)|^2 dV$ where dV is a Γ -invariant volume form. We have the obvious isomorphism $\mathcal{L}_\Gamma^2(\tilde{X}) \simeq \mathcal{L}^2(X)$. We may regard elements of $\mathcal{L}_\Gamma^2(\tilde{X})$ as functions $f(\gamma, x)$ as above which are constant in γ .

Elements $B \in \mathcal{B}_\Gamma$ with properly supported kernels, or kernels which decay fast enough off the diagonal, act on $\mathcal{L}_\Gamma^2(\tilde{X})$. Indeed, \mathcal{R} acts trivially on $\mathcal{L}_\Gamma^2(\tilde{X})$, so \mathcal{B}_Γ acts by the quotient

algebra $\mathcal{B}_\Gamma/\mathcal{R}_\Gamma$. We will be working with subalgebras of Toeplitz operators where the action is clearly well-defined.

Remark: Let us define $\mathcal{B}_\Gamma^\Gamma$ as the subalgebra of \mathcal{B}_Γ of elements which commute with both R_γ and L_γ for all γ . Then we have: $\mathcal{B}_\Gamma^\Gamma(\tilde{X}) \simeq \mathcal{B}(X)$. We may write the (Schwartz) kernel of an element of \mathcal{B} as $B(\gamma, x, \gamma', x')$. It belongs to \mathcal{B}_Γ if $B(\alpha\gamma, x, \alpha\gamma', x') = B(\gamma, x, \gamma', x')$ and it belongs to $\mathcal{B}_\Gamma^\Gamma$ if additionally $B(\gamma\alpha, x, \alpha, \gamma'\alpha, x') = B(\gamma, x, \gamma', x')$

We have been talking about algebras of bounded operators, but our main interest is in C^* algebras of Toeplitz operators. Everything we have said restricts to these subalgebras once we have defined the appropriate notions.

4.2. Toeplitz operators. The positive hermitian holomorphic line bundle $(L, h) \rightarrow M$ pulls back under π to one $(\tilde{L}, \tilde{h}) \rightarrow \tilde{M}$. This induces an inner product on the space $H^0(\tilde{M}, \tilde{L}^N)$ of entire holomorphic sections of \tilde{L}^N . We denote by $\mathcal{H}^2(\tilde{M}, \tilde{L}^N)$ the space of L^2 holomorphic sections relative to this inner product.

As in the quotient, there exists an associated S^1 bundle \tilde{X} with a contact (connection) form $\tilde{\alpha}$ such that

$$\begin{array}{ccc} \tilde{X} & \rightarrow & \tilde{M} \\ \downarrow & & \downarrow \\ X & \rightarrow & M \end{array}$$

commutes. The vertical arrows are covering maps and the horizontal ones are S^1 bundles. We denote the deck transformation group of $\tilde{X} \rightarrow X$ by $\tilde{\Gamma}$. It is isomorphic to Γ , so when no confusion is possible we drop the $\tilde{}$. Since all objects are lifted from quotients, it is clear that $\tilde{\Gamma}$ acts by contact transformations of $\tilde{\alpha}$. Let us denote operator of translation by γ on \tilde{M} by L_γ .

We denote by $\mathcal{H}^2(\tilde{X})$ the Hardy space of L^2 CR functions on \tilde{X} . They are boundary values of holomorphic functions in the strongly pseudoconvex complex manifold

$$(31) \quad \tilde{D}^* = \{(z, v) \in \tilde{L}^* : h_z(v) < 1\}$$

which are $\mathcal{L}^2(\tilde{X})$. The group $\tilde{\Gamma}$ acts on $\tilde{D}^* \subset \tilde{L}^*$ with quotient the compact disc bundle $D^* \subset L^* \rightarrow M$. In this setting it is known (cf. [Gromov-Henkin-Shubin (1998)], Theorem 0.2) that

$$\dim_\Gamma \mathcal{H}^2(\tilde{X}) = \infty.$$

Due to the S^1 symmetry, holomorphic functions on \tilde{D}^* are easily related to CR holomorphic functions on \tilde{X} . We denote by

$$(32) \quad \tilde{\Pi} : \mathcal{L}^2(\tilde{X}) \rightarrow \mathcal{H}^2(\tilde{X})$$

the Szegő (orthogonal) projection. Under the S^1 action, we have

$$(33) \quad \mathcal{H}^2(\tilde{X}) = \bigoplus_{N=1}^{\infty} \mathcal{H}_N^2(\tilde{X}), \quad \tilde{\Pi} = \bigoplus_{N=1}^{\infty} \tilde{\Pi}_N.$$

As on X , we have $\mathcal{H}_N^2(\tilde{X}) \simeq \mathcal{H}^2(\tilde{M}, \tilde{L}^N)$. We refer to [Gromov-Henkin-Shubin (1998)] (see example 2) on p. 559) for the proof that $\tilde{L} \rightarrow \tilde{M}$ has many holomorphic sections.

So far we have discussed L^2 functions on \tilde{X} . More important are periodic functions. We endow them with a Hilbert space structure by setting:

$$(34) \quad \begin{cases} \mathcal{L}_\Gamma^2(\tilde{X}) = \{f \in L_{loc}^2(\tilde{X}) : L_\gamma f = f\}, \\ \mathcal{H}_\Gamma^2(\tilde{X}) = \{f \in \mathcal{L}_\Gamma^2(\tilde{X}), \bar{\partial}_b f = 0\}, \end{cases}$$

with the inner product $\langle \cdot, \cdot \rangle_\Gamma$ obtained by integrating over a fundamental domain \mathcal{D} for Γ . Both are direct sums of weight spaces $\mathcal{L}_{\Gamma,N}^2$, resp. $\mathcal{H}_{\Gamma,N}^2(\tilde{X})$ for the S^1 action on \tilde{X} . There exists a Hilbert space isomorphism $L : \mathcal{H}_\Gamma^2(\tilde{X}) \rightarrow \mathcal{H}^2(X)$, namely by lifting $L\varphi = p^*\varphi$ under the covering map. The adjoint of L is given by:

$$L^*f = p_*(f1_{\mathcal{D}}).$$

We thus have

$$(35) \quad LL^* = Id : \mathcal{H}_\Gamma^2(\tilde{X}) \rightarrow \mathcal{H}_\Gamma^2(\tilde{X}).$$

We observe that the spaces $\mathcal{H}_{\Gamma,N}^2(\tilde{X})$ and $\mathcal{H}^2(\tilde{X})$ are completely unrelated and have different dimensions.

We now consider Toeplitz algebras. Since \tilde{X} is non-compact in general, we must take some care that Toeplitz operators are well-defined. Otherwise, the definitions are the same as for X : $\mathcal{T}^s(\tilde{X})$ is the space of operators of the form $\tilde{\Pi}A\tilde{\Pi}$ where $A \in \Psi_{S^1}^s(\tilde{X})$ is the space of properly supported pseudodifferential operators commuting with S^1 . We also define $\mathcal{T}^{-\infty}$ as the space of such $\tilde{\Pi}A\tilde{\Pi}$ with A having a smooth properly supported kernel.

We then distinguish the automorphic Toeplitz operators:

$$\mathcal{T}_\Gamma^*(\tilde{X}) = \{\tilde{\Pi}A\tilde{\Pi} \in \mathcal{T}^*(\tilde{X}) : L_\gamma^* \tilde{\Pi}A\tilde{\Pi} L_\gamma = \tilde{\Pi}A\tilde{\Pi}\}.$$

We note that $[L_\gamma, \tilde{\Pi}] = 0$ or equivalently $\tilde{\Pi}(\gamma x, \gamma y) = \tilde{\Pi}(x, y)$. So the operative condition is that $A \in \Psi_{S^1, \Gamma}$, the space of pseudodifferential operators commuting with Γ . The associated symbols $a_N(z, \bar{z})$ are exactly the Γ -invariant symbols on \tilde{M} . We note:

Proposition 4.1. *There exists a representation ρ_Γ of $\mathcal{T}_\Gamma^*(\tilde{X})$ on \mathcal{H}_Γ^2 preserving each $\mathcal{H}_{\Gamma,N}^2$.*

Proof. Since $L_\gamma \Pi A \Pi f = \Pi A \Pi f$ whenever $L_\gamma f = f$, the only issue is whether Af is well-defined for $f \in \mathcal{H}_\Gamma$. However, A is a polyhomogeneous sum of $a_j \mathcal{N}^{-j}$ where a_j is a periodic function, so the action is certainly defined. □

Let us denote by $\mathcal{K}_\Gamma = \ker \rho_\Gamma$. We further denote by $\rho_{\Gamma,N}$ the associated representation on $\mathcal{H}_{\Gamma,N}^2$, and put $\mathcal{K}_\Gamma = \ker \rho_{\Gamma,N}$.

Proposition 4.2. $\mathcal{K}_\Gamma \subset \mathcal{T}^{-\infty}(\tilde{X})$.

Proof. Assume that $\Pi A \Pi \in \mathcal{T}_\Gamma^*(\tilde{X})$ annihilates $\mathcal{H}_\Gamma^2(\tilde{X}) \simeq \mathcal{H}^2(X)$. This means that $\langle Af, g \rangle_{\mathcal{D}} = 0$ for all $f, g \in \mathcal{H}_\Gamma^2(\tilde{X})$. In particular it implies that the ‘Berezin symbol’ $\tilde{\Pi}_N a_N \tilde{\Pi}_N(z, z) = 0$. However, asymptotically, $N^{-m} \tilde{\Pi}_N a_N \tilde{\Pi}_N(z, z) \sim a_0(z)$ for a zeroth order Toeplitz operator. One sees by induction on the terms in (10) that $A \sim 0$. □

It could happen that $\mathcal{K}_\Gamma \neq 0$, unlike the analogous representation on $\mathcal{H}^2(X)$ which defines Toeplitz operators. For each N there could exist $a_N \in C^\infty(M)$ with $\|a_N\|_{L^2} = 1$ which is orthogonal to the finite dimensional space $\mathcal{H}_{\Gamma,N}^2(\tilde{X}) \simeq \mathcal{H}_N^2(X)$ but which is not orthogonal to $\mathcal{H}_N^2(\tilde{X})$. Then $T = \tilde{\Pi}_N a_N \tilde{\Pi}_N \in \mathcal{K}_{\Gamma,N}$.

We now relate $\mathcal{T}_\Gamma^*(\tilde{X})$ to $\mathcal{T}^*(X)$. In preparation, we relate the Szegő kernels on X, \tilde{X} . First, we consider a fixed N . The following is proved in [Shiffman-Zelditch]:

Proposition 4.3. *The degree N Szegő kernels of X, \tilde{X} are related by:*

$$\Pi_N(x, y) = \sum_{\gamma \in \Gamma} \tilde{\Pi}_N(\gamma \cdot x, y).$$

The same formula defines the Szegő projector $L_{\Gamma,N}^2 \rightarrow \mathcal{H}_{\Gamma,N}^2$.

The key point is to use the estimate

$$(36) \quad |\tilde{\Pi}_N(x, y)| \leq C e^{-\sqrt{N} \tilde{d}(x, y)}$$

where $\tilde{d}(x, y)$ is the Riemannian distance with respect to the Kähler metric $\tilde{\omega}$ to show that the sum converges for sufficiently large N . The estimates show that $\tilde{\Pi}_N$ acts on $\mathcal{L}^\infty(\tilde{X})$. Since $\mathcal{H}_\Gamma^2(\tilde{X}) \cap C(\tilde{X}) \subset \mathcal{L}^\infty(\tilde{X})$, $\tilde{\Pi}_N$ acts on $H_\Gamma^2(\tilde{X})$. The formula is an immediate consequence of writing

$$\tilde{\Pi}_N s_N(x) = \int_{\tilde{X}} \tilde{\Pi}_N(x, y) s_N(y) dV(y) = \int_{\mathcal{D}} \sum_{\gamma \in \Gamma} \tilde{\Pi}_N(x, \gamma y) s_N(y) dV(y).$$

for a periodic s_N in terms of a fundamental domain.

Now we consider the full Szegő kernel:

Corollary 4.4. *We have $L\Pi_N = \tilde{\Pi}_N L$ for each N as operators from $\mathcal{L}_N^2(X)$ to $\mathcal{H}_{\Gamma,N}^2$. Hence, $L\Pi = \tilde{\Pi} L : \mathcal{L}^2(X) \rightarrow \mathcal{H}_\Gamma^2(\tilde{X})$.*

The identity (30) for the larger algebra of bounded operators suggests that $\mathcal{T}_\Gamma^*(\tilde{X})$ should be a larger algebra than $\mathcal{T}^*(X)$. However, this is not the case.

Proposition 4.5. *L induces an algebra isomorphism $\mathcal{T}^*(X) \simeq \mathcal{T}_\Gamma(\tilde{X})/\mathcal{K}_\Gamma(\tilde{X})$.*

Proof. It follows from Proposition 4.1 and Corollary 4.4 that

$$(37) \quad L\Pi a \Pi = \tilde{\Pi} p^* a \tilde{\Pi} L : L^2(X) \rightarrow \mathcal{H}_\Gamma^2(\tilde{X}) \iff \Pi a \Pi = L^* \tilde{\Pi} p^* a \tilde{\Pi} L : \mathcal{L}^2(X) \rightarrow \mathcal{H}_\Gamma^2(X).$$

Equality on the designated spaces is equivalent to equality in the algebras. Further, the equality $LL^* = I : \mathcal{H}_\Gamma^2 \rightarrow \mathcal{H}_\Gamma^2$ implies that the linear isomorphism is an algebra isomorphism. \square

It may seem surprising that $\mathcal{T}_\Gamma(\tilde{X})$ is so ‘small’. We recall that one obtains $\mathcal{T}^*(X)$ by representing $\mathcal{T}_\Gamma(\tilde{X})$ on $\mathcal{H}_\Gamma^2(\tilde{X})$. When dealing with all bounded operators, the kernel is very large (\mathcal{R}_Γ) and it is also large if we fix N and consider the associated Toeplitz algebra. But the kernel is trivial if we consider the full Toeplitz algebra.

We also have:

Corollary 4.6. *L induces algebra isomorphisms $\mathcal{T}_N(X) \simeq \mathcal{T}_{\Gamma,N}(\tilde{X})/\mathcal{K}_{\Gamma,N}$.*

Now let us relate such automorphisms to automorphisms on \tilde{X} . The concrete identification with $\mathcal{T}(X)$ is by (37). We have:

Proposition 4.7. *There is a natural identification of:*

- automorphisms α on $\mathcal{T}^*(X)$ with automorphisms $\tilde{\alpha}$ on $\mathcal{T}_\Gamma^*(\tilde{X})/\mathcal{K}_\Gamma$;
- automorphisms α on $\mathcal{T}^*(X)/\mathcal{T}^{-\infty}(X)$ with automorphisms $\tilde{\alpha}$ on $\mathcal{T}_\Gamma^*(\tilde{X})/\mathcal{T}_\Gamma^{-\infty}$.

Proof. The first statement is clear since the algebras are the isomorphic. An automorphism of \mathcal{T}_Γ^* descends to \mathcal{T}^* if and only if it preserves \mathcal{K}_Γ . Concretely, we wish to set:

$$(38) \quad \alpha(\Pi a \Pi) = L^* \tilde{\alpha}(\tilde{\Pi} p^* a \tilde{\Pi}) L.$$

The inner operator must be determined by the left side for this to be well-defined. We have

$$(39) \quad \tilde{\Pi} p^* a \tilde{\Pi} \equiv L \Pi a \Pi L^* \mod \mathcal{K}_\Gamma.$$

The same equivalence is true modulo the larger subalgebra $\mathcal{T}_\Gamma^{-\infty}$. Since $\mathcal{T}_\Gamma^{-\infty}/\mathcal{K}_\Gamma = \mathcal{T}^{-\infty}(X)$, the second statement is correct. \square

4.3. Completion of the proof of Theorem 1.6. We now prove statements (iii)-(iv) of the Theorem. The following gives an ‘upper bound’ on existence of semi-classical automorphisms.

Lemma 4.8. *Suppose that α is an order-preserving automorphism of $\mathcal{T}^*(X)/\mathcal{T}^{-\infty}$. Let $\tilde{\alpha}$ denote the corresponding automorphisms of $\mathcal{T}_\Gamma^*(\tilde{X})/\mathcal{K}_\Gamma$. Then there exists a canonical transformation χ of (M, ω) and a unitary Toeplitz quantum map \tilde{U}_χ on $H^2(\tilde{X})$ such that*

$$\tilde{\alpha}(P) = \tilde{U}_\chi^* P \tilde{U}_\chi,$$

and such that

$$L_\gamma^{-1} \tilde{U}_\chi L_\gamma = M_\gamma \tilde{U}_\chi,$$

where $M_\gamma \in \mathcal{T}_\Gamma^*(\tilde{M})'$.

Proof. By Lemma(3.1), we know that $\tilde{\alpha}$ is given by conjugation by a Toeplitz quantum map \tilde{U}_χ . We may define M_γ by the formula above since \tilde{U}_χ is invertible. We then determine its properties.

We have:

$$L_\gamma^{-1} \tilde{U}_\chi^{-1} L_\gamma P L_\gamma^{-1} \tilde{U}_\chi L_\gamma = \tilde{U}_\chi^{-1} P \tilde{U}_\chi, \quad \forall P \in \mathcal{T}_\Gamma^*(\tilde{M}).$$

Hence,

$$M_\gamma^{-1} \tilde{U}_\chi^{-1} P \tilde{U}_\chi M_\gamma = \tilde{U}_\chi^{-1} P \tilde{U}_\chi \iff M_\gamma^{-1} P M_\gamma = P \quad \forall P \in \mathcal{T}_\Gamma^*(\tilde{M}).$$

It follows that M_γ is central. This proves (iii) of Theorem 1.6. \square

The ‘lower bound’ is given by:

Lemma 4.9. *Suppose that χ is a symplectic map of (M, ω) . Then it lifts to a contact transformation $\tilde{\chi}$ of $(\tilde{X}, \tilde{\alpha})$. The associated quantum map $U_{\tilde{\chi}}$ on \tilde{X} defines (by conjugation) an order preserving automorphism $\tilde{\alpha}$ of $\mathcal{T}_\Gamma(\tilde{X})$ which descends to an automorphism of $\mathcal{T}^*(X)/\mathcal{T}^{-\infty}$. If $\tilde{\alpha}$ preserves \mathcal{K}_Γ , then it defines an automorphism of all of $\mathcal{T}^*(X)$.*

Proof. χ automatically lifts to \tilde{M} as a symplectic map commuting with the action of Γ . By Proposition 2.3 it lifts to \tilde{X} as a contact transformation. We then define a unitary quantum map by

$$\tilde{U}_{N\tilde{\chi}} = \tilde{\Pi}_N \sigma T_{\tilde{\chi}} \tilde{\Pi}_N,$$

where σ is a function on \tilde{M} which makes the operator unitary on $\mathcal{H}^2(\tilde{X})$. See [Zelditch (1997)] for background.

A crucial issue now is the commutation relations between $\tilde{U}_{N\tilde{\chi}}$ and $\tilde{\Gamma}$. When χ lifts to X , i.e. is quantizable, then $\tilde{\chi}$ commutes with $\tilde{\Gamma}$. In this case, χ is quantizable as a quantum map and there was no need to lift it to \tilde{X} to quantize it as an automorphism.

Assume however that χ does not lift to X and consider the commutation relations of the translation by $\tilde{\chi}$ with left translations by elements of $\tilde{\Gamma}$. The commutator $\tilde{\chi}\tilde{\gamma}\tilde{\chi}^{-1}\tilde{\gamma}^{-1}$ covers the identity map of \tilde{M} since the lift of χ to \tilde{M} commutes with Γ . Furthermore, it commutes with the S^1 action on \tilde{X} . It follows that

$$(40) \quad \tilde{\chi}\tilde{\gamma}\tilde{\chi}^{-1}\tilde{\gamma}^{-1} = T_{e^{i\theta_{\gamma,\chi}}}$$

where the right side is translation by the element $e^{i\theta_{\gamma,\chi}}$. The angle $\theta_{\gamma,\chi}$ is apriori a function on \tilde{M} . However, the left side is a contact transformation covering the identity and therefore $d\theta = d\theta + d\theta_{\gamma,\chi}$, i.e. $\theta_{\gamma,\chi}$ is a constant.

After quantizing, the same commutator identity holds for the operators. Therefore the operators M_γ are central. It follows that the automorphism

$$(41) \quad \tilde{\alpha}(P) = \tilde{U}_\chi^* P \tilde{U}_\chi$$

satisfies

$$\tilde{\alpha}_N(\tilde{\Pi}_N p^* a_N \tilde{\Pi}_N) \in \mathcal{T}_\Gamma^*(\tilde{X}).$$

It therefore descends to $\mathcal{T}^*(X)/\mathcal{T}^{-\infty}(X)$ by (38), i.e. as

$$\tilde{\alpha}_N(\tilde{\Pi}_N p^* a_N \tilde{\Pi}_N) = L^* \tilde{U}_\chi^* \tilde{\Pi}_N p^* a_N \tilde{\Pi}_N \tilde{U}_\chi L.$$

Unitarity of L then implies that α is also an automorphism. If the automorphism preserves \mathcal{K}_Γ then it also descends to $\mathcal{T}^*(X)$. □

An obvious question is whether the condition that the automorphism preserve \mathcal{K}_Γ is equivalent to the quantization condition that χ lift to X . Clearly, quantizability in the sense of Definition 1.5 implies preservation of \mathcal{K}_Γ , since the quantum map $U_{\chi,N}$ is well defined on the spaces \mathcal{H}_N . The converse is not obvious, since we only know apriori that the automorphism induces automorphisms α_N of the finite rank observables $Op_N(a_N)$ for fixed N . Abstractly, such automorphisms must be given by conjugations by unitary operators on \mathcal{H}_N , but it is not clear that these unitary operators are Toeplitz quantum maps in the sense of 1.5.

We end the section with

4.3.1. Proof of Corollary 1.7.

Proof. This follows immediately from (ii) if $H^1(M, \mathbb{C}) = \emptyset$. If $H^1(M, \mathbb{C}) \neq \emptyset$, then by Theorem 1.6 (iii) there exists a symplectic map χ of M and a Toeplitz Fourier integral operator \tilde{V}_χ on \tilde{M} and a central operator M_γ such that $T_\gamma^* \tilde{V}_\chi T_\gamma = M_\gamma \tilde{V}_\chi$, and such that α is induced by conjugation by \tilde{V} . But by definition, α is also given by conjugation by U . Now the Schwarz

kernel of U , hence U , lifts to a Γ invariant kernel \tilde{U} on \tilde{M} . By assumption, $\tilde{U}A\tilde{U}^*$ has the same complete symbol as $\tilde{V}A\tilde{V}^*$ for any Toeplitz operator A on M , lifted to \mathcal{T}_Γ . By Lemma 3.2, it follows that $\tilde{U} = \tilde{V} + R$, where R is a smoothing Toeplitz operator. It follows that $M_\gamma = 1$ (hence $\theta_{\gamma, X} = 1$ for all γ), and therefore the symplectic map $\tilde{\chi}$ underlying \tilde{V} , when lifted to \tilde{X} , is invariant under the deck transformation group $\tilde{\Gamma}$ of $\tilde{X} \rightarrow X$. \square

5. QUANTIZATION OF TORUS MAPS

To clarify the issues involved, we consider some standard examples on the symplectic $2m$ -torus $\mathbf{T}^{2m} = \mathbb{C}^m / \mathbb{Z}^{2nm}$. Since $H_1(\mathbf{T}^{2m}, \mathbb{C}) = \mathbb{C}^{2m}$, there will exist symplectic maps which cannot be quantized in the sense of Definitions 1.4 and 1.5 (a) as quantum maps, though they can and will be quantized as automorphisms. In fact, the distinction can already be illustrated with the simplest maps:

- Kronecker translations $T_\theta(x) = x + \theta$ ($x, \theta \in \mathbf{T}^{2m}$).
- Symplectic automorphisms $A \in Sp(2m, \mathbb{Z})$.

We begin by describing the line bundle on the torus and its universal cover. We follow [Zelditch (1997), Bleher-Shiffman-Zelditch(2001)] and refer there for further discussion.

The quotient setting is $L \rightarrow \mathbf{T}^{2n}$, where L is the bundle with curvature $\sum_j dz_j \wedge d\bar{z}_j$. Sections of L^N are theta-functions of level N . On the universal cover, we have the pulled back bundle $L_{\mathbf{H}} = \mathbb{C} \times \mathbb{C}^m \rightarrow \mathbb{C}^m$. Its associated principal S^1 bundle $\mathbb{C}^m \times S^1 \rightarrow \mathbb{C}^m$ is the *reduced Heisenberg group* $\mathbf{H}_{\text{red}}^m$. We recall that it is the quotient under the subgroup $(0, \mathbb{Z})$ in the center of the simply connected Heisenberg group $\mathbf{H}^m = \mathbb{C}^m \times \mathbb{R}$ with group law

$$(\zeta, t) \cdot (\eta, s) = (\zeta + \eta, t + s + \frac{1}{2}\Im(\zeta \cdot \bar{\eta})).$$

The identity element is $(0, 0)$ and $(\zeta, t)^{-1} = (-\zeta, -t)$. The reduced Heisenberg group is thus $\mathbf{H}_{\text{red}}^m = \mathbf{H}^m / \{(0, k) : k \in \mathbb{Z}\} = \mathbb{C}^m \times S^1$ with group law

$$(\zeta, e^{2\pi i t}) \cdot (\eta, e^{2\pi i s}) = (\zeta + \eta, e^{2\pi i [t+s+\frac{1}{2}\Im(\zeta \cdot \bar{\eta})]}).$$

We now equip $\mathbf{H}_{\text{red}}^m$ with the left-invariant connection form

$$(42) \quad \alpha^L = \frac{1}{2} \sum_q (\xi_q dx_q - x_q d\xi_q) - \frac{dt}{2\pi}, \quad (\zeta = x + i\xi),$$

whose curvature equals the symplectic form $\omega = \sum_q dx_q \wedge d\xi_q$. The kernel of α^L is the distribution of horizontal planes. To define the Szegő kernel we further need to split the complexified horizontal spaces into their holomorphic and anti-holomorphic parts. The left-invariant (CR-) holomorphic (respectively anti-holomorphic) vector fields Z_q^L (respectively \bar{Z}_q^L) on $\mathbf{H}_{\text{red}}^m$ are the horizontal lifts of the vector fields $\frac{\partial}{\partial z_q}$, respectively $\frac{\partial}{\partial \bar{z}_q}$ with respect to α^L . We then define the Hardy space $\mathcal{H}^2(\mathbf{H}_{\text{red}}^m)$ of CR holomorphic functions to be the functions in $\mathcal{L}^2(\mathbf{H}_{\text{red}}^m)$ satisfying the left-invariant Cauchy-Riemann equations $\bar{Z}_q^L f = 0$ ($1 \leq q \leq m$) on $\mathbf{H}_{\text{red}}^m$. For $N = 1, 2, \dots$, we further define $\mathcal{H}_N^2 \subset \mathcal{H}^2(\mathbf{H}_{\text{red}}^m)$ as the (infinite-dimensional) Hilbert space of square-integrable CR functions f such that $f \circ r_\theta = e^{iN\theta} f$ as before. The representation \mathcal{H}_1^2 is irreducible and may be identified with the Bargmann-Fock space of

entire holomorphic functions on \mathbb{C}^n which are square integrable relative to $e^{-|z|^2}$. The Szegő kernel $\Pi_N^{\mathbf{H}}(x, y)$ is the orthogonal projection to \mathcal{H}_N^2 . It is given by

$$(43) \quad \Pi_N^{\mathbf{H}}(x, y) = \frac{1}{\pi^m} N^m e^{iN(t-s)} e^{N(\zeta \cdot \bar{\eta} - \frac{1}{2}|\zeta|^2 - \frac{1}{2}|\eta|^2)}, \quad x = (\zeta, t), \quad y = (\eta, s).$$

We note that it satisfies the estimates in (36). In this model example, Proposition 4.3 was proven in [Zelditch (1997)].

Finally, we describe the circle bundle X in the quotient setting. The lattice \mathbb{Z}^{2m} may be embedded as a subgroup $\mathbf{H}_{\mathbb{Z}}^m$ of $\mathbf{H}_{\mathbb{R}}^m$ under the homomorphism

$$(44) \quad \iota(m, n) = (m, n, e^{i\pi m \cdot n}).$$

We will denote the image by $\mathbf{H}_{\mathbb{Z}}^m$. To clarify the role of the factors of $\frac{1}{2}$ we show that $\mathbf{H}_{\mathbb{Z}}^m$ is indeed a subgroup:

$$(45) \quad \begin{aligned} (m, n, e^{i\pi m \cdot n}) \cdot (m', n', e^{i\pi m' \cdot n'}) &= (m + m', n + n', e^{i\pi(m \cdot n + m' \cdot n' + mn' - m'n)}) \\ &= (m + m', n + n', e^{i\pi(m \cdot n + m' \cdot n' + mn' + m'n)}) \\ &= (m + m', n + n', e^{i\pi((m+m') \cdot (n+n'))}). \end{aligned}$$

We then put: $X = H_{\mathbb{Z}}^m \backslash H_{\mathbb{R}}^m m_{\mathbb{R}}$, i.e. X is the *left* quotient of $H_{\mathbb{R}}^m$ by $\mathbf{H}_{\mathbb{Z}}^m$. The left-invariant contact form α^L descends to X as a contact form and a connection form for the principal S^1 bundle $X \rightarrow \mathbb{C}^m / \mathbb{Z}^{2m}$.

5.1. Kronecker translations. We first show that irrational Kronecker translations

$$(46) \quad T_{(a,b)} f(x, \xi) = f(x + a, \xi + b)$$

are non-quantizable as Toeplitz quantum maps.

Proposition 5.1. *$T_{(a,b)}$ fails to be quantizable for all $(a, b) \in \mathbb{R}^{2n} / \mathbb{Z}^{2n}$. $T_{(a,b)}$ is quantizable at level N iff $Na, Nb \in \mathbb{Z}^{2n}$.*

Proof. By Proposition 2.2, the map lifts if and only if translations preserve holonomy of homologically non-trivial loops. The loops on \mathbf{T}^{2m} that we need to consider are given in local coordinates by $\gamma_{m,n}(t) = (tm, tn)$. Horizontal lifts to X are given by $\tilde{\gamma}_{m,n}(t) = (tm, tn, 1)$. At $t = 1$ we obtain $(m, n, 1) \sim (0, 0, e^{i\pi m \cdot n})$. Hence, the holonomy of the path $\gamma_{m,n}$ equals $e^{i\pi m \cdot n}$.

Now translate the loop $\gamma_{(m,n)}$ by (a, b) to obtain the loop $\gamma_1(t) = (tm + a, tn + b)$. A horizontal lift to X is given by $\tilde{\gamma}_1(t) = (tm + a, tn + b, e^{\pi i(b \cdot m - a \cdot n)t})$. It is the projection to X of the left translate by $(a, b, 0)$ of the original horizontal path. At $t = 1$ the endpoint is $(m + a, n + b, e^{\pi i(b \cdot m - a \cdot n)}) = (a, b, e^{i\pi[m \cdot n + 2(b \cdot m - a \cdot n)]})$. Hence, the holonomy changed by $e^{2\pi i(b \cdot m - a \cdot n)}$. The holonomy is preserved iff $b \cdot m - a \cdot n \in \mathbb{Z}$ for all $(m, n) \in \mathbb{Z}^{2m}$ iff $(a, b) \in \mathbb{Z}^{2m}$.

Lifting to level N means changing the holonomy to $e^{2\pi i N \theta_{\gamma}}$. So the condition to lift becomes $(a, b) \in \frac{1}{N} \mathbb{Z}^2$

□

Remark: The non-quantizability of $T_{(a,b)}$ is due to the left/right invariance of various objects. $T_{(a,b)}$ only lifts to $H_{\mathbb{Z}} \backslash H_{\mathbb{R}}$ as right translation by $(a, b, 0)$. But right translation by an element of $H_{\mathbb{R}}$ does not preserve the left invariant contact form. Equivalently, $T_{(a,b)}$ only lifts to a

contact transformation of $H_{\mathbb{R}}$ if it lifts to left translation by $(a, b, 0)$. But then the lift does not descend to $H_{\mathbb{Z}} \backslash H_{\mathbb{R}}$.

5.1.1. *Kronecker translations as automorphisms.* It is easy to see that Kronecker translations define automorphisms of the relevant algebras. We lift $T_{(a,b)}$ to $H_{\mathbb{R}}^n$ as the contact transformation of left multiplication

$$T'_{(a,b)}(x, \xi, e^{2\pi i t}) := (x + a, \xi + b, e^{2\pi i t} e^{\pi i(a\xi - bx)}).$$

Although the map $T_{(a,b)}$ does not descend to the quotient as a map, we claim:

Proposition 5.2. *Kronecker maps $T_{(a,b)}$ have the following properties:*

- (i) $T_{(a,b)}$ defines an automorphism of \mathcal{T}_{Γ}^* .
- (ii) $T_{(a,b)}$ defines an automorphism of $\mathcal{T}^{\infty}(X)/\mathcal{T}^{-\infty}(X)$ by

$$\alpha_{a,b,N}(Op_N(a)) = Op_N(a \circ T_{(a,b)}).$$

- (iii) However, $\alpha_{(a,b)}$ does not preserve \mathcal{K}_{Γ} and does not define an automorphism of $\mathcal{T}^{\infty}(X)$.

Proof. (i) Left translation by (a, b) defines an automorphism of \mathcal{T}_{Γ}^* because, by (43), the Szegő kernel commutes with left translations, i.e.

$$(47) \quad \Pi_N^H(\alpha \cdot x, \alpha \cdot y) = \Pi_N^H(x, y) \quad \forall \alpha.$$

Indeed, it is the kernel of a convolution operator.

(ii)

Consider the conjugates $T_{\gamma}^{-1} T'_{a,b} T_{\gamma}$ where $\gamma \in \Gamma = \mathbb{Z}^{2m}$. An easy computation shows that

$$T_{\gamma}^{-1} T'_{a,b} T_{\gamma} f(x, \xi, t) = M_{\gamma} T'_{a,b} f(x), \quad \text{where } M_{\gamma} f(x, \xi, t) = f(x, \xi, t + \omega(\gamma, (a, b))).$$

We need to show that M_{γ} commutes with every Toeplitz operator $\tilde{\Pi}_N \sigma \tilde{\Pi}_N$ with symbol lifted from $\mathbb{C}^m / \mathbb{Z}^{2m}$. Since the symbol is invariant under the central circle, it is sufficient to show that $[M_{\gamma}, \tilde{\Pi}_N] = 0$. But this follows as long as $\tilde{\Pi}_N(z \cdot x, z \cdot y) = \tilde{\Pi}_N(x, y)$ for any $z = e^{it} \in S^1$, the center of the Heisenberg group. But this follows because

$$\tilde{\Pi}_N(z \cdot x, z \cdot y) = |z|^{2N} \tilde{\Pi}_N(x, y) = \tilde{\Pi}_N(x, y).$$

Since left translation commutes with $\tilde{\Pi}_N$, the automorphism descends to the quotient as:

$$\begin{aligned} \alpha_N(\Pi_N a_N \Pi_N) &= L^* T'_{\alpha,\beta}^* (\tilde{\Pi}_N p^* a_N \tilde{\Pi}_N) T'_{\alpha,\beta} L \\ &= L^* \tilde{\Pi}_N (T'_{a,b} p^* a_N) \tilde{\Pi}_N L. \end{aligned}$$

This is the stated formula.

(iii) A Kronecker automorphisms $\alpha_{(a,b)}$ can only preserve $\mathcal{K}_{\Gamma,N}$ if $\Pi_N a \Pi_N = 0$ implies $\Pi_N(a \circ T_{a,b}) \Pi_N = 0$. But if this were the case, the elements $\Pi_N e^{i\langle k, x \rangle} \Pi_N$ would be distinct eigenoperators with eigenvalues $e^{i\langle k, (a,b) \rangle}$. This contradicts the finite dimensionality of the algebra for fixed N .

□

5.2. Quantum cat maps. We now show, in a similar way, that symplectic linear maps of \mathbf{T}^2 always define quantum automorphisms, even though they do not always define quantum maps. We write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and define $g(x, \xi) = (ax + b\xi, cx + d\xi)$ on the torus. It lifts to the reduced Heisenberg group by $g(x, \xi, t) = (g(x, \xi), t)$.

Proposition 5.3. *T_g is quantizable iff $a \cdot c, b \cdot d \in 2\mathbb{Z}$.*

Proof. We go through the same calculation as for Kronecker translations. This time, the horizontal lift of the transformed loop is $(t(a \cdot m + b \cdot n), t(c \cdot m + d \cdot n), 1)$. At $t = 1$ the endpoint is

$$((a \cdot m + b \cdot n), (c \cdot m + d \cdot n), 1) = (0, 0, e^{i\pi(a \cdot m + b \cdot n)(c \cdot m + d \cdot n)}).$$

Since the holonomy of the original path was $e^{i\pi m \cdot n}$, the change in holonomy equals

$$e^{i\pi(m \cdot n - (a \cdot m + b \cdot n)(c \cdot m + d \cdot n))} = 1 \iff ac, bd \in 2\mathbb{Z}.$$

Here we use that $ad + bc \equiv 1 \pmod{2\mathbb{Z}}$. □

5.2.1. Linear maps as automorphisms. It is known that quantizable linear maps (cat maps) on the quotient define quantum maps with exact Egorov theorems (see e.g. [Zelditch (1997)]). We now show that non-quantizable maps as well defined automorphisms by the exact Egorov formula:

Proposition 5.4. *T_g defines an automorphism of $\mathcal{T}^\infty(X)/\mathcal{T}^{-\infty}(X)$ by*

$$\alpha_{g;N}(Op_N(a)) = Op_N(a \circ T_g).$$

Proof. Consider the conjugates $T_\gamma^{-1}T_gT_\gamma$ where $\gamma \in \Gamma$. We have:

$$T_\gamma^{-1}T_gT_\gamma f(x, \xi, t) = M_\gamma T_g f(x), \quad \text{where } M_\gamma f(x, \xi, t) = f((x, \xi) + (I - g)\gamma, t + \omega(\gamma, z) - \omega(\gamma, g(z + \gamma))).$$

M_γ is the composition of translation $T_{(I-g)\gamma}$ with a central translation. Since $(I - g)\gamma$ is in the lattice, translation by this element commutes with left invariant operators. Thus, $M_\gamma \in \mathcal{T}_\Gamma^*(\tilde{X})'$.

Thus, the automorphism descends to the quotient as:

$$\begin{aligned} \alpha_N(\Pi_N a_N \Pi_N) &= L^* T_g^* (\tilde{\Pi}_N p^* a_N \tilde{\Pi}_N) T_g L \\ &= L^* \tilde{\Pi}_N (T_g p^* a_N) \tilde{\Pi}_N L. \end{aligned}$$

□

6. SPECTRA OF AUTOMORPHISMS

In this article, our interest lies in the automorphisms defined by symplectic maps. But most of the interest in quantizations of quantizable symplectic maps, at least in the physics literature, is in their spectral theory as unitary operators $U_{\chi, N}$ on the finite dimensional Hilbert spaces $\mathcal{H}_N(X)$. In this section, we point out how the most important aspects of this spectral theory of U pertain only to the spectrum of the associated automorphism UAU^* . The main point is that the reformulation suggests generalizations to other kinds of automorphisms. We also tie together the automorphisms of Toeplitz algebras on the torus with the well-known ones on the rotation algebra.

6.1. Spectra of automorphisms of Hilbert-Schmidt algebras. We let \mathcal{H} denote a Hilbert space, and denote by \mathcal{HS} the algebra of Hilbert-Schmidt operators on \mathcal{H} , i.e. the operators for which the inner product $\langle A, B \rangle := \text{Tr} AB^*$ is finite. We let $*$ denote the adjoint on \mathcal{H} . A finite dimensional algebra of Hilbert-Schmidt operators is of course a full matrix algebra, and its automorphisms are given by conjugation by unitary operators.

Suppose that α is an automorphism of \mathcal{HS} .

Definition 6.1. We say that an automorphism α of \mathcal{HS} is:

- a $*$ -automorphism if: $\alpha(A^*) = \alpha(A)^*$.
- unitary if: $\langle \alpha(A), \alpha(B) \rangle = \langle A, B \rangle$.
- a conjugation if: there exists a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}$ s.th. $\alpha(A) = UAU^*$.

We also say that the automorphism is *tracial* if: $\text{Tr} \alpha(A) = \text{Tr} A$ for all A of trace class.

We will consider the eigenvalues and ‘eigenoperators’ of a unitary $*$ -automorphism on \mathcal{HS} :

$$\alpha(A) = e^{i\theta} A.$$

If α is a unitary automorphism, then (as a unitary operator) it possesses an orthonormal basis of eigenoperators $\{A_j\}$.

The following is elementary from the definitions.

Proposition 6.2. We have:

- (i) A tracial $*$ -automorphism is unitary.
- (ii) The composition of any two eigenoperators is a (possibly zero) eigenoperator.
- (iii) If A_j is an eigenoperator, then $A_j^* A_j$ is an invariant operator, i.e. $\alpha(A_j^* A_j) = A_j^* A_j$.

Proof. (i) Immediate from the fact that

$$\langle \alpha(A), \alpha(B) \rangle = \text{Tr} \alpha(A) \alpha(B)^* = \text{Tr} \alpha(AB^*) = \text{Tr} AB^* = \langle A, B \rangle.$$

(ii) - (iii) These statements follow from the equations $\alpha(A_j A_k) = \alpha(A_j) \alpha(A_k) = e^{i(\theta_j + \theta_k)} A_j A_k$, and $\alpha(A_j^*) = [\alpha(A_j)]^* = e^{-i\theta_j} \alpha(A_j)$. □

In the case $\alpha(A) = U^* A U$ we note that the eigenoperators of the automorphism are given by

$$(48) \quad \alpha(\varphi_{N,j} \otimes \varphi_{N,k}^*) = e^{i(\theta_{N,j} - \theta_{N,k})} (\varphi_{N,j} \otimes \varphi_{N,k}^*),$$

where $\{(\varphi_{N,j}, e^{i\theta_{N,j}})\}$ are the spectral data of U .

6.2. Spectral problems of quantum chaos. The main problems on quantum maps pertain to the spacings between eigenvalues (the pair correlation problem) and the asymptotics of matrix elements relative to eigenfunctions of the operators.

6.2.1. Pair correlation problem. Let us recall that the pair correlation function ρ_{2N} (PCF) of a quantum map $\{U_N\}$ with Planck constant $1/N$ is the function on \mathbb{R} defined by

$$\int_{\mathbb{R}} f(x) d\rho_{2N}(x) = \sum_{\ell=0}^{\infty} \hat{f}\left(\frac{\ell}{N}\right) |\text{Tr} U_N^\ell|^2.$$

Its limit as $N \rightarrow \infty$, when one exists, is the PCF of the quantum map. Clearly, knowledge of $d\rho_2^N$ is equivalent to knowledge of its *form factor* $|Tr U_N^\ell|^2$. We observe that the form factor depends only on the automorphism:

Proposition 6.3. *The form factor of a conjugation automorphism $\{\alpha_N\}$ is given by*

$$Tr \alpha_N^\ell = \sum_j \langle \alpha_N^\ell A_j, A_j \rangle$$

where $\{A_j\}$ is an orthonormal basis for $\mathcal{H}S_N$.

Indeed, if $\alpha(A)_N = U_N^* A U_N$, then $\alpha_N^\ell = U_N^\ell \otimes U_N^{-\ell}$ on $\mathcal{H}S_N$, and we have $Tr_{\mathcal{H}S_N} U_N^\ell \otimes U_N^{-\ell} = |Tr_{\mathcal{H}S_N} U_N^\ell|^2$.

Hence, the pair correlation problem makes sense for all unitary automorphisms α_N .

Problem Given any unitary automorphism α_N , determine $\frac{1}{d_N} \sum_{j=1}^{d_N^2} \delta(N(\vartheta_{N,j}))$ as $N \rightarrow \infty$, where

$$(49) \quad \alpha_N(\Phi_j) = e^{i\vartheta_{N,j}} \Phi_j$$

is the eigenvalue problem for the automorphism.

6.2.2. Problems of quantum ergodicity/mixing. We observe that these too can be formulated for any sequence of automorphisms. We rewrite the asymptotics of matrix elements $\langle A\varphi_{N,j}, \varphi_{N,k} \rangle = Tr A\varphi_{N,j} \otimes \varphi_{N,k}^*$ as the inner products $\langle A, \Phi_{jk} \rangle$. We observe that the eigenfunctions $\Phi_{k,k} = \varphi_{N,k} \otimes \varphi_{N,k}^*$ always have eigenvalue 1, i.e. they are invariant states of the automorphism.

It is simple to check that the proof of quantum ergodicity for quantizations of ergodic quantizable symplectic maps χ (see [Zelditch (1997)]) uses only the automorphism involved. It states that if a symplectic map χ is ergodic and quantizable, then the invariant states of the corresponding automorphism of the Toeplitz algebra are asymptotic to the traces $\tau_N(A) = \frac{1}{\dim \mathcal{H}_M} Tr A|_{\mathcal{H}_N}$.

It might be interesting to find generalizations of this result to other kinds of automorphisms.

6.3. Spectral theory of model automorphisms. We now point out that the automorphisms induced by model quantum maps on the torus are the same as the well-known automorphism of the finite dimensional rotation algebras.

6.3.1. The rotation algebra modulo N . We denote by G_N the finite Heisenberg group of order N^2 , generated by two elements U, V satisfying

$$U_2 U_1 = e^{2\pi i/N} U_1 U_2, \quad U_1^N = U_2^N = I,$$

and its group algebra by \mathcal{R}_N . G_N has a unique irreducible unitary representation ρ_N on \mathbb{C}^N given as follows: If we regard $\mathbb{C}^N = L^2(\mathbb{Z}/N\mathbb{Z})$ then

$$\rho_N(U_1)\psi(Q) = e^{\frac{2\pi i Q}{N}} \psi(Q), \quad \rho_N(U_2)\psi(Q) = \psi(Q+1).$$

Recall that the rotation algebra or non-commutative torus A_θ is the (pre-) C^* algebra generated by unitaries U_1, U_2 satisfying the Weyl commutation relation

$$U_2 U_1 = e^{2\pi i \alpha} U_1 U_2.$$

When $\alpha = 1/N$, A_θ has a large center generated by U_1^N, U_2^N . \mathcal{R}_N is obtained from A_θ by viewing central elements as scalars.

6.3.2. Toeplitz algebra and rotation algebra. We identify the rotation algebra with $\theta = \frac{2\pi}{N}$ to the algebra of Toeplitz operators $\Pi_N a \Pi_N$ on the torus. As verified by S. Nonnenmacher [Nonnenmacher (2003)], the elements

$$(50) \quad U_1 = e^{\pi N^2} \Pi_N e^{i\theta_1} \Pi_N, \quad U_2 = e^{\pi N^2} \Pi_N e^{i\theta_2} \Pi_N,$$

satisfy $U_j^N = I$. Here, $(e^{i\theta_1}, e^{i\theta_2})$ are the standard coordinates on the torus. Hence, any quantum map on the torus defines an automorphism of \mathcal{R}_N . Thus, we can identify the

automorphisms $\alpha_{g,N}$ quantizing $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the well-known automorphisms of \mathcal{R}^N

defined by $U_1 \rightarrow U_1^a U_2^b$, $U_2 \rightarrow U_1^c U_2^d$ (see e.g. [Narnhofer (1997)]).

We can also see easily from this point of view that Kronecker maps $T_{u,v}$ cannot in general be quantized as automorphisms. Namely, the quantization on \mathbb{R}^2 translates the symbol, so it would descend to $U_1 \rightarrow e^{iu} U_1$, $U_2 \rightarrow e^{iv} U_2$. To be well-defined, one needs e^{iu}, e^{iv} to be N th roots of unity, which of course they are not in the irrational case.

7. APPENDIX

The key elements of the Toeplitz algebra and its automorphisms are the $*$ product (13) and the Egorov formula (1). The purpose of this appendix is to direct the reader's attention to the existence of routine calculations of the complete symbols of compositions $a_N *_N b_N$ of symbols and of conjugations $U_N O p_N(a) U_N^*$ of observables by Toeplitz quantum maps. The method is to use the Boutet de Monvel - Sjöstrand parametrix for the Szegő kernel [Boutet de Monvel-Sjöstrand (1976)] as in [Zelditch (1998)]. Since the original version of this paper was written, several papers [Karabegov-Schlichenmaier (2001), Schlichenmaier (1999), Schlichenmaier (1998), Schlichenmaier (1999b), Shiffman-Tate-Zelditch (2003)] have also used this method to describe the $*$ product on symbols, so we will be brief.

Proposition 7.1. *Let (M, ω) be a compact kahler manifold. Then: The $*$ product defines an algebra structure on classical symbols. There exists an asymptotic expansion:*

$$\hat{f}_1 * \hat{f}_2(z, \bar{z}) \sim \sum_{k=0}^{\infty} N^{-k} B_k(f_1, f_2)$$

where $B_0(a, b) = f_1 \cdot f_2$, $B_1(f_2, f_2) = \frac{1}{2}\{f_1, f_2\}$ and where B_k is a bi-differential operator of $C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$.

Proof. Using the Boutet de Monvel-Sjöstrand parametrix as in [Zelditch (1997), Shiffman-Tate-Zelditch (2003)], one can obtain a complete asymptotic expansion of the covariant symbol $\Pi_N a \Pi_N b \Pi_N(z, \bar{z})$. One writes out $\Pi_N(z, w)$ as an oscillatory integral and applies complex stationary phase. For calculations of this kind we refer to [Shiffman-Tate-Zelditch (2003)].

To obtain $a *_N b$, we invert the Berezin transform I_N on symbols, as described in [Reshetikhin-Takhtajan (1990)] and elsewhere. It is invertible on formal power series, and the same inverse is well-defined on symbol expansions. Thus,

$$a *_N b \sim I_N^{-1} \Pi_N a \Pi_N b \Pi_N(z, \bar{z}).$$

This produces the symbol expansion claimed in the proposition. □

Proposition 7.2. *Let $U_{\chi,N}$ be a Toeplitz quantum map as in Definition 1.5. Then for any observable $\Pi_N a_N \Pi_N = Op_N(a_N)$, the contravariant symbol a_χ of $U_{\chi,N}^* Op_N(a_N) U_{\chi,N}$ possesses a complete asymptotic expansion*

$$a_\chi(z) \sim \sum_{k=0}^{\infty} N^{-k} V_k(a \circ \chi)$$

where $V_0(a) = a$, and where V_k is a differential operator of order at most $2k$.

Proof. As above, the expansion is obtained from the covariant symbol $U_{\chi,N}^* Op_N(a_N) U_{\chi,N}(z, \bar{z})$ by inverting the Berezin transform. The asymptotics of the covariant symbol follow by applying stationary phase to the oscillatory integral formula for $U_{\chi,N}^* Op_N(a_N) U_{\chi,N}$. □

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